

# ON THE $L^2$ -BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS AND THEIR COMMUTATORS WITH SYMBOLS IN $\alpha$ -MODULATION SPACES

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## 1. INTRODUCTION

Since the theory of pseudo-differential operators was established in 1970's, the  $L^2$ -boundedness of them with symbols in the Hörmander class  $S_{\rho,\delta}^0$  has been well investigated by many authors. Among them, Calderón-Vaillancourt [5] first treated the boundedness for the class  $S_{0,0}^0$ , which means that the boundedness of all the derivatives of symbols assures the  $L^2$ -boundedness of the corresponding operators. It should be mentioned that the boundedness of all the derivatives of symbols is not necessary in their proof. Being motivated by this argument, many authors as Coifman-Meyer [6], Cordes [8], Kato [17], Miyachi [19], Muramatu [20], Nagase [21] contributed to know the minimal assumption on the regularity of symbols for the corresponding operators to be  $L^2$ -bounded. They said that the boundedness of the derivatives of symbols up to a certain order, which exceeds  $n/2$ , assures the  $L^2(\mathbb{R}^n)$ -boundedness. Especially, Sugimoto [24] showed that symbols in the Besov space  $B_{(n/2, n/2)}^{(\infty, \infty), (1, 1)}$  implies the  $L^2$ -boundedness.

In the last decade, new developments in this problem have appeared. Sjöstrand [22] introduced a wider class than  $S_{0,0}^0$  which assures the  $L^2$ -boundedness and is now recognized as a special case of modulation spaces introduced by Feichtinger [9, 10, 11]. These spaces are based on the idea of quantum mechanics or time-frequency analysis. Sjöstrand class can be written as  $M^{\infty, 1}$  if we follow the notation of modulation spaces. Gröchenig-Heil [16] and Toft [26] gave some related results to Sjöstrand's one by developing the theory of modulation spaces. Boulkhemir [3] treated the same discussion for Fourier integral operators.

We remark that the relation between Besov and modulation spaces is well studied by the works of Gröbner [15], Toft [26] and Sugimoto-Tomita [25], and we know that the spaces  $B_{(n/2, n/2)}^{(\infty, \infty), (1, 1)}$  and  $M^{\infty, 1}$  have no inclusion relation with each others (see Appendix) although the class  $S_{0,0}^0$  is properly included in both spaces. In this sense, the results of Sugimoto [24] and Sjöstrand [22] are independent extension of Calderon-Vaillancourt's result.

The objective of this paper is to show that these two results, which appeared to be independent ones, can be proved based on the same principle. Especially we give another proof to Sjöstrand's result following the same argument used to prove Sugimoto's result. For the purpose, we use the notation of  $\alpha$ -modulation spaces  $M_{s,\alpha}^{p,q}$  ( $0 \leq \alpha \leq 1$ ), a parameterized family of function spaces, which includes Besov spaces  $B_s^{p,q}$  and modulation spaces  $M^{p,q}$  as special cases corresponding to

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$\alpha = 1$  and  $\alpha = 0$ . The  $\alpha$ -modulation spaces were introduced by Gröbner [15], and developed by the works of Feichtinger-Gröbner [12], Borup-Nielsen [1, 2] and Fornasier [13].

The following is our main result:

**Theorem 1.1.** *Let  $0 \leq \alpha \leq 1$ . Then there exists a constant  $C > 0$  such that*

$$\|\sigma(X, D)f\|_{L^2} \leq C\|\sigma\|_{M_{(\alpha n/2, \alpha n/2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \|f\|_{L^2}$$

for all  $\sigma \in M_{(\alpha n/2, \alpha n/2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $f \in \mathcal{S}(\mathbb{R}^n)$ .

The exact definition of the product  $\alpha$ -modulation space  $M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}$  will be given in Section 2, and the proof will be given in Section 3. Theorem 1.1 with  $\alpha = 1$  is the result of Sugimoto [24] while  $\alpha = 0$  Sjöstrand [22].

As an important application of Theorem 1.1, we can discuss the  $L^2$ -boundedness of the commutator  $[T, a]$  of the operator  $T$  and a Lipschitz function  $a(x)$ . Calderón [4] considered this problem when  $T$  is a singular integral operator of convolution type, and Coifman-Meyer [7] extended this argument to the case when  $T$  is a pseudo-differential operator with the symbol in the class  $S_{1,0}^1$ . Furthermore, Marschall [18] showed the  $L^2$ -boundedness of this commutator when the symbol is of the class  $S_{\rho, \delta}^m$  with  $m = \rho$ , especially the class  $S_{0,0}^0$ . On account of Theorem 1.1, it is natural to expect the same boundedness for symbols in Besov and modulation spaces. In fact we have the following theorem:

**Theorem 1.2.** *Let  $0 \leq \alpha \leq 1$ . Then there exists a constant  $C > 0$  such that*

$$\|[\sigma(X, D), a]f\|_{L^2} \leq C\|\nabla a\|_{L^\infty} \|\sigma\|_{M_{(\alpha n/2, \alpha n+1), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \|f\|_{L^2}$$

for all Lipschitz functions  $a$ ,  $\sigma \in M_{(\alpha n/2, \alpha n+1), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $f \in \mathcal{S}(\mathbb{R}^n)$ .

Theorem 1.2 with  $\alpha = 1$ , which requires  $\sigma \in B_{(n/2, n+1)}^{(\infty, \infty), (1, 1)}$ , is an extension of the result by Marschall [18] which treated the case  $\sigma \in B_{(r, N)}^{(\infty, \infty), (\infty, \infty)}$  with  $r > n/2$  and  $N > n + 1$ . Theorem 1.2 with  $\alpha = 0$  is a result of new type in this problem. The proof of Theorem 1.2 will be given in Section 4.

## 2. PRELIMINARIES

Let  $\mathcal{S}(\mathbb{R}^n)$  and  $\mathcal{S}'(\mathbb{R}^n)$  be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform  $\mathcal{F}f$  and the inverse Fourier transform  $\mathcal{F}^{-1}f$  of  $f \in \mathcal{S}(\mathbb{R}^n)$  by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

Let  $\sigma(x, \xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ . We denote by  $\mathcal{F}_1\sigma(y, \xi)$  and  $\mathcal{F}_2\sigma(x, \eta)$  the partial Fourier transform of  $\sigma$  in the first variable and in the second variable, respectively. That is,  $\mathcal{F}_1\sigma(y, \xi) = \mathcal{F}[\sigma(\cdot, \xi)](y)$  and  $\mathcal{F}_2\sigma(x, \eta) = \mathcal{F}[\sigma(x, \cdot)](\eta)$ . We also denote by  $\mathcal{F}_1^{-1}\sigma$  and  $\mathcal{F}_2^{-1}\sigma$  the partial inverse Fourier transform of  $\sigma$  in the first variable and in the second variable, respectively. We write  $\mathcal{F}_{1,2} = \mathcal{F}_1\mathcal{F}_2$  and  $\mathcal{F}_{1,2}^{-1} = \mathcal{F}_1^{-1}\mathcal{F}_2^{-1}$ , and note that  $\mathcal{F}_{1,2}$  and  $\mathcal{F}_{1,2}^{-1}$  are the usual Fourier transform and inverse Fourier transform of functions on  $\mathbb{R}^n \times \mathbb{R}^n$ .

We introduce the  $\alpha$ -modulation spaces based on Borup-Nielsen [1, 2]. Let  $B(\xi, r)$  be the ball with center  $\xi$  and radius  $r$ , where  $\xi \in \mathbb{R}^n$  and  $r > 0$ . A countable set  $\mathcal{Q}$

of subsets  $Q \subset \mathbb{R}^n$  is called an admissible covering if  $\mathbb{R}^n = \cup_{Q \in \mathcal{Q}} Q$  and there exists a constant  $n_0$  such that  $\#\{Q' \in \mathcal{Q} : Q \cap Q' \neq \emptyset\} \leq n_0$  for all  $Q \in \mathcal{Q}$ . We denote by  $|Q|$  the Lebesgue measure of  $Q$ , and set  $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ , where  $\xi \in \mathbb{R}^n$ . Let  $0 \leq \alpha \leq 1$ ,

$$(2.1) \quad \begin{aligned} r_Q &= \sup\{r > 0 : B(c_r, r) \subset Q \text{ for some } c_r \in \mathbb{R}^n\}, \\ R_Q &= \inf\{R > 0 : Q \subset B(c_R, R) \text{ for some } c_R \in \mathbb{R}^n\}. \end{aligned}$$

We say that an admissible covering  $\mathcal{Q}$  is an  $\alpha$ -covering of  $\mathbb{R}^n$  if  $|Q| \asymp \langle \xi \rangle^{\alpha n}$  (uniformly) for all  $\xi \in Q$  and  $Q \in \mathcal{Q}$ , and there exists a constant  $K \geq 1$  such that  $R_Q/r_Q \leq K$  for all  $Q \in \mathcal{Q}$ , where “ $|Q| \asymp \langle \xi \rangle^{\alpha n}$  (uniformly) for all  $\xi \in Q$  and  $Q \in \mathcal{Q}$ ” means that there exists a constant  $C > 0$  such that

$$C^{-1} \langle \xi \rangle^{\alpha n} \leq |Q| \leq C \langle \xi \rangle^{\alpha n} \quad \text{for all } \xi \in Q \text{ and } Q \in \mathcal{Q}.$$

Let  $r_Q$  and  $R_Q$  be as in (2.1). We note that

$$(2.2) \quad B(c_Q, r_Q/2) \subset Q \subset B(d_Q, 2R_Q) \quad \text{for some } c_Q, d_Q \in \mathbb{R}^n,$$

and there exists a constant  $\kappa > 0$  such that

$$(2.3) \quad |Q| \geq \kappa \quad \text{for all } Q \in \mathcal{Q}$$

since  $|Q| \asymp \langle \xi_Q \rangle^{\alpha n} \geq 1$ , where  $\xi_Q \in Q$ . By (2.1), we see that  $s_n r_Q^n \leq |Q| \leq s_n R_Q^n$ , where  $s_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . This implies

$$s_n \leq \frac{|Q|}{r_Q^n} = \frac{R_Q^n}{r_Q^n} \frac{|Q|}{R_Q^n} \leq K^n \frac{|Q|}{R_Q^n} \leq K^n s_n,$$

that is,

$$(2.4) \quad |Q| \asymp r_Q^n \asymp R_Q^n \quad \text{for all } Q \in \mathcal{Q}$$

(see [1, Appendix B]). We frequently use the fact

$$(2.5) \quad \langle \xi_Q \rangle \asymp \langle \xi'_Q \rangle \quad \text{for all } \xi_Q, \xi'_Q \in Q \text{ and } Q \in \mathcal{Q}.$$

If  $\alpha \neq 0$ , then (2.5) follows directly from the definition of  $\alpha$ -covering  $|Q| \asymp \langle \xi_Q \rangle^{\alpha n}$ . By (2.4), if  $\alpha = 0$  then  $R_Q^n \asymp |Q| \asymp \langle \xi_Q \rangle^{\alpha n} = 1$ , and consequently there exists  $R > 0$  such that  $R_Q \leq R$  for all  $Q \in \mathcal{Q}$ . Hence, by (2.2), we have  $Q \subset B(d_Q, 2R)$  for some  $d_Q \in \mathbb{R}^n$ . This implies that (2.5) is true even if  $\alpha = 0$ .

Given an  $\alpha$ -covering  $\mathcal{Q}$  of  $\mathbb{R}^n$ , we say that  $\{\psi_Q\}_{Q \in \mathcal{Q}}$  is a corresponding bounded admissible partition of unity (BAPU) if  $\{\psi_Q\}_{Q \in \mathcal{Q}}$  satisfies

- (1)  $\text{supp } \psi_Q \subset Q$ ,
- (2)  $\sum_{Q \in \mathcal{Q}} \psi_Q(\xi) = 1$  for all  $\xi \in \mathbb{R}^n$ ,
- (3)  $\sup_{Q \in \mathcal{Q}} \|\mathcal{F}^{-1} \psi_Q\|_{L^1} < \infty$ .

We remark that an  $\alpha$ -covering  $\mathcal{Q}$  of  $\mathbb{R}^n$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$  actually exists for every  $0 \leq \alpha \leq 1$  ([1, Proposition A.1]). Let  $1 \leq p, q \leq \infty$ ,  $s \in \mathbb{R}$ ,  $0 \leq \alpha \leq 1$  and  $\mathcal{Q}$  be an  $\alpha$ -covering of  $\mathbb{R}^n$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$ . Fix a sequence  $\{\xi_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{R}^n$  satisfying  $\xi_Q \in Q$  for every  $Q \in \mathcal{Q}$ . Then the  $\alpha$ -modulation space  $M_{s, \alpha}^{p, q}(\mathbb{R}^n)$  consists of all  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\|f\|_{M_{s, \alpha}^{p, q}} = \left( \sum_{Q \in \mathcal{Q}} \langle \xi_Q \rangle^{sq} \|\psi_Q(D)f\|_{L^p}^q \right)^{1/q} < \infty,$$

where  $\psi(D)f = \mathcal{F}^{-1}[\psi \widehat{f}] = (\mathcal{F}^{-1}\psi) * f$ . We remark that the definition of  $M_{s,\alpha}^{p,q}$  is independent of the choice of the  $\alpha$ -covering  $\mathcal{Q}$ , BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}}$  and sequence  $\{\xi_Q\}_{Q \in \mathcal{Q}}$  (see [2, Section 2]). Let  $\psi \in \mathcal{S}(\mathbb{R}^n)$  be such that

$$(2.6) \quad \text{supp } \psi \subset [-1, 1]^n, \quad \sum_{k \in \mathbb{Z}^n} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.$$

If  $\alpha = 0$  then the  $\alpha$ -modulation space  $M_{s,\alpha}^{p,q}(\mathbb{R}^n)$  coincides with the modulation space  $M_s^{p,q}(\mathbb{R}^n)$ , that is,  $\|f\|_{M_{s,\alpha}^{p,q}} \asymp \|f\|_{M_s^{p,q}}$ , where

$$\|f\|_{M_s^{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\psi(D - k)f\|_{L^p}^q \right)^{1/q}.$$

If  $s = 0$  we write  $M^{p,q}(\mathbb{R}^n)$  instead of  $M_0^{p,q}(\mathbb{R}^n)$ . Let  $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$  be such that

$$(2.7) \quad \text{supp } \varphi_0 \subset \{|\xi| \leq 2\}, \quad \text{supp } \varphi \subset \{1/2 \leq |\xi| \leq 2\}, \quad \varphi_0(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) = 1$$

for all  $\xi \in \mathbb{R}^n$ . Set  $\varphi_j(\xi) = \varphi(\xi/2^j)$  if  $j \geq 1$ . On the other hand, if  $\alpha = 1$  then the  $\alpha$ -modulation space  $M_{s,\alpha}^{p,q}(\mathbb{R}^n)$  coincides with the Besov space  $B_s^{p,q}(\mathbb{R}^n)$ , that is,  $\|f\|_{M_{s,\alpha}^{p,q}} \asymp \|f\|_{B_s^{p,q}}$ , where

$$\|f\|_{B_s^{p,q}} = \left( \sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(D)f\|_{L^p}^q \right)^{1/q}.$$

We remark that we can actually check that the  $\alpha$ -covering  $\mathcal{Q}$  with the corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$  given in [1, Proposition A.1] (see Lemma 4.3) satisfies

$$(2.8) \quad \sum_{Q \in \mathcal{Q}} \psi_Q(D)f = f \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \quad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n)$$

and

$$(2.9) \quad \sum_{Q, Q' \in \mathcal{Q}} \psi_Q(Dx) \psi_{Q'}(D\xi) \sigma(x, \xi) = \sigma(x, \xi) \quad \text{in } \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$$

for all  $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ , where  $0 \leq \alpha < 1$ ,

$$\psi_Q(Dx) \psi_{Q'}(D\xi) \sigma = \mathcal{F}_{1,2}^{-1}[(\psi_Q \otimes \psi_{Q'}) \mathcal{F}_{1,2} \sigma] = [(\mathcal{F}^{-1} \psi_Q) \otimes (\mathcal{F}^{-1} \psi_{Q'})] * \sigma$$

and  $\psi_Q \otimes \psi_{Q'}(x, \xi) = \psi_Q(x) \psi_{Q'}(\xi)$ . In the case  $\alpha = 1$ , (2.8) and (2.9) are well known facts, since we can take  $\{\varphi_j\}_{j \geq 0}$  as a BAPU corresponding to the  $\alpha$ -covering  $\{\{|\xi| \leq 2\}, \{\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}\}_{j \geq 1}\}$ , where  $\{\varphi_j\}_{j \geq 0}$  is as in (2.7). In the rest of this paper, we assume that an  $\alpha$ -covering  $\mathcal{Q}$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$  always satisfies (2.8) and (2.9).

We introduce the product  $\alpha$ -modulation space  $M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$  as a symbol class of pseudo-differential operators. Let  $s_1, s_2 \in \mathbb{R}$ ,  $0 \leq \alpha \leq 1$  and  $\mathcal{Q}$  be an  $\alpha$ -covering of  $\mathbb{R}^n$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$ . Fix two sequences  $\{x_Q\}_{Q \in \mathcal{Q}}, \{\xi_{Q'}\}_{Q' \in \mathcal{Q}} \subset \mathbb{R}^n$  satisfying  $x_Q \in Q$  and  $\xi_{Q'} \in Q'$  for every  $Q, Q' \in \mathcal{Q}$ . Then the product  $\alpha$ -modulation space  $M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$  consists of all  $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$  such that

$$\|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} = \sum_{Q \in \mathcal{Q}} \sum_{Q' \in \mathcal{Q}} \langle x_Q \rangle^{s_1} \langle \xi_{Q'} \rangle^{s_2} \|\psi_Q(Dx) \psi_{Q'}(D\xi) \sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} < \infty.$$

We note that  $M_{(0,0),(0,0)}^{(\infty,\infty),(1,1)}(\mathbb{R}^n \times \mathbb{R}^n) = M^{\infty,1}(\mathbb{R}^{2n})$ , since we can take  $\{\psi(\cdot - k)\}_{k \in \mathbb{Z}^n}$  as a BAPU corresponding to the  $\alpha$ -covering  $\{k + [-1, 1]^n\}_{k \in \mathbb{Z}^n}$ , and  $\psi \otimes \psi$  satisfies (2.6) with  $2n$  instead of  $n$ , where  $\alpha = 0$  and  $\psi \in \mathcal{S}(\mathbb{R}^n)$  is as in (2.6). Similarly,  $M_{(s_1,s_2),(1,1)}^{(\infty,\infty),(1,1)}(\mathbb{R}^n \times \mathbb{R}^n) = B_{(s_1,s_2)}^{(\infty,\infty),(1,1)}(\mathbb{R}^n \times \mathbb{R}^n)$ , where

$$\|\sigma\|_{B_{(s_1,s_2)}^{(\infty,\infty),(1,1)}} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{js_1 + ks_2} \|\varphi_j(D_x) \varphi_k(D_\xi) \sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}$$

and  $\{\varphi_j\}_{j \geq 0}, \{\varphi_k\}_{k \geq 0}$  are as in (2.7) (see Sugimoto [24, p.116]).

We shall end this section by showing the following basic properties of an  $\alpha$ -covering:

**Lemma 2.1.** *Let  $\mathcal{Q}$  be an  $\alpha$ -covering of  $\mathbb{R}^n$  and  $R > 0$ . Then the following are true:*

- (1) *If  $(Q + B(0, R)) \cap Q' \neq \emptyset$ , then there exists a constant  $\kappa > 0$  such that*

$$\kappa^{-1} \langle \xi_Q \rangle \leq \langle \xi_{Q,Q'} \rangle \leq \kappa \langle \xi_Q \rangle \quad \text{and} \quad \kappa^{-1} \langle \xi_{Q'} \rangle \leq \langle \xi_{Q,Q'} \rangle \leq \kappa \langle \xi_{Q'} \rangle$$

*for all  $\xi_Q \in Q$ ,  $\xi_{Q'} \in Q'$  and  $\xi_{Q,Q'} \in (Q + B(0, R)) \cap Q'$ , where  $\kappa$  is independent of  $Q, Q'$ . In particular,  $\langle \xi_Q \rangle \asymp \langle \xi_{Q'} \rangle$ .*

- (2) *There exists a constant  $n'_0$  such that*

$$\#\{Q' \in \mathcal{Q} : (Q + B(0, R)) \cap Q' \neq \emptyset\} \leq n'_0 \quad \text{for all } Q \in \mathcal{Q}.$$

*Proof.* Assume that  $(Q + B(0, R)) \cap Q' \neq \emptyset$ , where  $Q, Q' \in \mathcal{Q}$ .

We consider the first part. Let  $\xi_{Q,Q'} \in (Q + B(0, R)) \cap Q'$ . Since  $\xi_{Q,Q'} = \widetilde{\xi_Q} + \xi$  for some  $\widetilde{\xi_Q} \in Q$  and  $\xi \in B(0, R)$ , we see that  $\langle \xi_{Q,Q'} \rangle \asymp \langle \widetilde{\xi_Q} \rangle$ . Hence, by (2.5),  $\langle \xi_Q \rangle \asymp \langle \widetilde{\xi_Q} \rangle \asymp \langle \xi_{Q,Q'} \rangle$ . Similarly,  $\langle \xi_{Q'} \rangle \asymp \langle \xi_{Q,Q'} \rangle$ .

We next consider the second part. It follows from the first part that  $|Q| \asymp \langle \xi_Q \rangle^{\alpha n} \asymp \langle \xi_{Q'} \rangle^{\alpha n} \asymp |Q'|$ , and consequently

$$(2.10) \quad |Q| \asymp |Q'| \quad \text{if } (Q + B(0, R)) \cap Q' \neq \emptyset.$$

Let  $B(c_Q, r_Q/2) \subset Q \subset B(d_Q, 2R_Q)$  and  $B(c_{Q'}, r_{Q'}/2) \subset Q' \subset B(d_{Q'}, 2R_{Q'})$ , where  $Q, Q' \in \mathcal{Q}$  (see (2.2)). By (2.3), (2.4) and (2.10), we see that  $R_Q \asymp R_{Q'}$  and  $R_Q \geq \kappa_1$  for some constant  $\kappa_1$  independent of  $Q \in \mathcal{Q}$ . Then

$$\begin{aligned} \emptyset \neq (Q + B(0, R)) \cap Q' &\subset (B(d_Q, 2R_Q) \cap B(0, R)) \cap B(d_{Q'}, 2R_{Q'}) \\ &= B(d_Q, 2R_Q + R) \cap B(d_{Q'}, 2R_{Q'}) \subset B(d_Q, (2 + \kappa_1^{-1}R)R_Q) \cap B(d_{Q'}, 2R_{Q'}). \end{aligned}$$

Combining  $B(d_Q, (2 + \kappa_1^{-1}R)R_Q) \cap B(d_{Q'}, 2R_{Q'}) \neq \emptyset$  and  $R_Q \asymp R_{Q'}$ , we obtain that  $B(d_{Q'}, 2R_{Q'}) \subset B(d_Q, \kappa_2 R_Q)$  for some constant  $\kappa_2 \geq 2$  independent of  $Q, Q'$ . Hence, since  $c_Q \in B(d_Q, \kappa_2 R_Q)$  and  $r_Q \asymp R_Q$ , if  $(Q + B(0, R)) \cap Q' \neq \emptyset$  then

$$(2.11) \quad Q' \subset B(d_{Q'}, 2R_{Q'}) \subset B(d_Q, \kappa_2 R_Q) \subset B(c_Q, \kappa_3 r_Q),$$

where  $\kappa_3$  is independent of  $Q, Q' \in \mathcal{Q}$ . Let  $\mathcal{Q}_i$ ,  $i = 1, \dots, n_0$ , be subsets of  $\mathcal{Q}$  such that  $\mathcal{Q} = \bigcup_{i=1}^{n_0} \mathcal{Q}_i$  and the elements of  $\mathcal{Q}_i$  are pairwise disjoint (see [1, Lemma B.1]). Set  $A_Q = \{Q' \in \mathcal{Q} : (Q + B(0, R)) \cap Q' \neq \emptyset\}$ . By (2.11), we have

$$\sum_{Q' \in A_Q \cap \mathcal{Q}_i} |Q'| \leq |B(c_Q, \kappa_3 r_Q)| = (2\kappa_3)^n |B(c_Q, r_Q/2)| \leq (2\kappa_3)^n |Q|$$

for all  $1 \leq i \leq n_0$ . Therefore, by (2.10), we see that

$$(\#A_Q)|Q| \leq \sum_{i=1}^{n_0} \sum_{Q' \in A_Q \cap \mathcal{Q}_i} (\kappa_4|Q'|) \leq \kappa_4 \sum_{i=1}^{n_0} (2\kappa_3)^n |Q| = n_0(2\kappa_3)^n \kappa_4 |Q|,$$

that is,  $\#A_Q \leq n_0(2\kappa_3)^n \kappa_4$ . The proof is complete.  $\square$

### 3. PSEUDO-DIFFERENTIAL OPERATORS AND $\alpha$ -MODULATION SPACES

In this section, we prove Theorem 1.1. For  $\sigma \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ , the pseudo-differential operator  $\sigma(X, D)$  is defined by

$$\sigma(X, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) d\xi \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

In order to prove Theorem 1.1, we prepare the following lemmas:

**Lemma 3.1** ([24, Lemma 2.2.1]). *There exists a pair of functions  $\varphi, \chi \in \mathcal{S}(\mathbb{R}^n)$  satisfying*

- (1)  $\int_{\mathbb{R}^n} \varphi(\xi) \chi(\xi) d\xi = 1$ ,
- (2)  $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : |\xi| < 1\}$  and  $\text{supp } \widehat{\chi} \subset \{\eta \in \mathbb{R}^n : |\eta| < 1\}$ .

**Lemma 3.2** ([24, Lemma 2.2.2]). *Let  $g_\tau(x) = g(x, \tau)$  be such that*

- (1)  $g(x, \tau) \in L^2(\mathbb{R}_x^n \times \mathbb{R}_\tau^n)$ ,
- (2)  $\sup_{x \in \mathbb{R}^n} \|g(x, \cdot)\|_{L^1(\mathbb{R}^n)} < \infty$ ,
- (3)  $\text{supp } \widehat{g_\tau} \subset \Omega$ ,

where  $\widehat{g_\tau}(y) = \mathcal{F}_1 g(y, \tau)$  and  $\Omega$  is a compact subset of  $\mathbb{R}^n$  independent of  $\tau$ . If  $h(x) = \int_{\mathbb{R}^n} e^{ix \cdot \tau} g(x, \tau) d\tau$ , then there exists a constant  $C > 0$  such that

$$\|h\|_{L^2} \leq C|\Omega|^{1/2} \|g\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)},$$

where  $C$  is independent of  $g$  and  $\Omega$ .

**Lemma 3.3** ([24, Lemma 2.2.3]). *Let  $\sigma_x(\xi) = \sigma(x, \xi)$  be such that*

- (1)  $\sigma_x(\xi) \in L^1(\mathbb{R}_\xi^n) \cap L^2(\mathbb{R}_\xi^n)$ ,
- (2)  $\text{supp } \widehat{\sigma_x} \subset \Omega$ ,

where  $\widehat{\sigma_x}(\eta) = \mathcal{F}_2 \sigma(x, \eta)$  and  $\Omega$  is a compact subset of  $\mathbb{R}^n$  independent of  $x$ . Then there exists a constant  $C > 0$  such that

$$\|\sigma(X, D)f\|_{L^2} \leq C|\Omega|^{1/2} \sup_{x \in \mathbb{R}^n} \|\sigma(x, \cdot)\|_{L^2} \|f\|_{L^2}$$

for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , where  $C$  is independent of  $\sigma$  and  $\Omega$ .

**Lemma 3.4.** *Let  $0 \leq \alpha \leq 1$ ,  $s_1, s_2 \in \mathbb{R}$  and  $\sigma \in M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$ . Then there exists a family  $\{\sigma_\epsilon\}_{0 < \epsilon < 1} \subset \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  such that*

- (1)  $\langle \sigma(X, D)f, g \rangle = \lim_{\epsilon \rightarrow 0} \langle \sigma_\epsilon(X, D)f, g \rangle$  for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$ ,
- (2)  $\|\sigma_\epsilon\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \leq C \|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}}$  for all  $0 < \epsilon < 1$ ,

where  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{S}' \times \mathcal{S}}$  and  $C$  is independent of  $\sigma$ .

*Proof.* Let  $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$  be such that  $\varphi(0) = 1$ ,  $\text{supp } \widehat{\varphi} \subset \{|y| < 1\}$ ,  $\int_{\mathbb{R}^n} \psi(x) dx = 1$ . Set  $\Phi(x, \xi) = \varphi(x) \varphi(\xi)$ ,  $\Psi(x, \xi) = \psi(x) \psi(\xi)$  and

$$\sigma_\epsilon(x, \xi) = \Phi_\epsilon(x, \xi) (\Psi_\epsilon * \sigma)(x, \xi),$$

where  $\Phi_\epsilon(x, \xi) = \Phi(\epsilon x, \epsilon \xi)$  and  $\Psi_\epsilon(x, \xi) = \epsilon^{-2n} \Psi(x/\epsilon, \xi/\epsilon)$ . Note that  $\sigma_\epsilon \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ ,  $\Phi(0, 0) = 1$  and  $\int_{\mathbb{R}^{2n}} \Psi(x, \xi) dx d\xi = 1$ . Then the well known fact  $\sigma_\epsilon \rightarrow \sigma$  in  $\mathcal{S}'(\mathbb{R}^{2n})$  as  $\epsilon \rightarrow 0$  implies (1).

Let us consider (2). If

$$(3.1) \quad \|\Phi_\epsilon \sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \leq C \|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \quad \text{for all } 0 < \epsilon < 1$$

and

$$(3.2) \quad \|\Psi_\epsilon * \sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \leq C \|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \quad \text{for all } 0 < \epsilon < 1,$$

then

$$\|\Phi_\epsilon(\Psi_\epsilon * \sigma)\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \leq C \|\Psi_\epsilon * \sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \leq C \|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}}$$

for all  $0 < \epsilon < 1$ , and this is the desired estimate. Let us prove (3.1) and (3.2). But, (3.2) is trivial since

$$\begin{aligned} \|\Psi_\epsilon * \sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} &= \sum_{Q, Q' \in \mathcal{Q}} \langle x_Q \rangle^{s_1} \langle \xi_{Q'} \rangle^{s_2} \|\psi_Q(D_x) \psi_{Q'}(D_\xi) (\Psi_\epsilon * \sigma)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &= \sum_{Q, Q' \in \mathcal{Q}} \langle x_Q \rangle^{s_1} \langle \xi_{Q'} \rangle^{s_2} \|\Psi_\epsilon * (\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &\leq \sum_{Q, Q' \in \mathcal{Q}} \langle x_Q \rangle^{s_1} \langle \xi_{Q'} \rangle^{s_2} \|\Psi_\epsilon\|_{L^1(\mathbb{R}^n \times \mathbb{R}^n)} \|\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}, \end{aligned}$$

where  $\mathcal{Q}$  is an  $\alpha$ -covering with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$ . We prove (3.1). Noting

$$\text{supp } \mathcal{F}_{1,2} \Phi_\epsilon \subset \{(y, \eta) : |y| < \epsilon, |\eta| < \epsilon\} \subset \{(y, \eta) : |y| < 1, |\eta| < 1\}$$

for all  $0 < \epsilon < 1$ , we see that

$$\text{supp } \mathcal{F}_{1,2} [\Phi_\epsilon \psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma] \subset \{(y, \eta) : y \in Q + B(0, 1), \eta \in Q' + B(0, 1)\}.$$

Since  $\sup_{Q \in \mathcal{Q}} \|\mathcal{F}^{-1} \psi_Q\|_{L^1} < \infty$ , we have by (2.9) and Lemma 2.1

$$\begin{aligned} \|\Phi_\epsilon \sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} &= \sum_{\tilde{Q}, \tilde{Q}' \in \mathcal{Q}} \langle x_{\tilde{Q}} \rangle^{s_1} \langle \xi_{\tilde{Q}'} \rangle^{s_2} \|\psi_{\tilde{Q}}(D_x) \psi_{\tilde{Q}'}(D_\xi) (\Phi_\epsilon \sigma)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &\leq \sum_{\tilde{Q}, \tilde{Q}' \in \mathcal{Q}} \sum_{Q, Q' \in \mathcal{Q}} \\ &\quad \times \langle x_{\tilde{Q}} \rangle^{s_1} \langle \xi_{\tilde{Q}'} \rangle^{s_2} \|\psi_{\tilde{Q}}(D_x) \psi_{\tilde{Q}'}(D_\xi) [\Phi_\epsilon (\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma)]\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &= \sum_{Q, Q' \in \mathcal{Q}} \sum_{\substack{\tilde{Q} \in \mathcal{Q} \\ \tilde{Q} \cap (Q + B(0, 1)) \neq \emptyset}} \sum_{\substack{\tilde{Q}' \in \mathcal{Q} \\ \tilde{Q}' \cap (Q' + B(0, 1)) \neq \emptyset}} \\ &\quad \times \langle x_{\tilde{Q}} \rangle^{s_1} \langle \xi_{\tilde{Q}'} \rangle^{s_2} \|\psi_{\tilde{Q}}(D_x) \psi_{\tilde{Q}'}(D_\xi) [\Phi_\epsilon (\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma)]\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &\leq C(n'_0)^2 \sum_{Q, Q' \in \mathcal{Q}} \langle x_Q \rangle^{s_1} \langle \xi_{Q'} \rangle^{s_2} \|\Phi_\epsilon (\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &\leq C(n'_0)^2 \|\Phi\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|\sigma\|_{M_{(s_1, s_2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}}, \end{aligned}$$

where  $n'_0$  is as in Lemma 2.1 (2). The proof is complete.  $\square$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Lemma 3.4, it is enough to prove Theorem 1.1 with  $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ . Let  $\varphi, \chi$  be as in Lemma 3.1,  $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $f \in \mathcal{S}(\mathbb{R}^n)$ . By Lemma 3.1, we have

$$\begin{aligned}
 h(x) &= \sigma(X, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) d\xi \\
 (3.3) \quad &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) \left( \int_{\mathbb{R}^n} (\varphi \chi)(\xi - \tau) d\tau \right) d\xi \\
 &= \int_{\mathbb{R}^n} e^{ix \cdot \tau} \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi + \tau) \varphi(\xi) \chi(\xi) \widehat{f}(\xi + \tau) d\xi \right) d\tau.
 \end{aligned}$$

Let  $0 \leq \alpha \leq 1$  and  $\mathcal{Q}$  be an  $\alpha$ -covering with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$ . Set

$$\sigma_\tau(x, \xi) = \sigma(x, \xi + \tau) \quad \text{and} \quad f_\tau = \mathcal{F}^{-1}[\varphi \widehat{f}(\cdot + \tau)].$$

Then, by (2.9),

$$(3.4) \quad \sigma_\tau(x, \xi) = \sum_{Q, Q' \in \mathcal{Q}} [\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma_\tau](x, \xi) = \sum_{Q, Q' \in \mathcal{Q}} \sigma_{\tau, Q, Q'}(x, \xi),$$

where

$$\sigma_{\tau, Q, Q'}(x, \xi) = [\psi_Q(D_x) \psi_{Q'}(D_\xi) \sigma_\tau](x, \xi).$$

Note that  $\sigma_{\tau, Q, Q'}(x, \xi) \in \mathcal{S}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ . By (3.3) and (3.4),

$$\begin{aligned}
 h(x) &= \int_{\mathbb{R}^n} e^{ix \cdot \tau} \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma_\tau(x, \xi) \chi(\xi) \left( \varphi(\xi) \widehat{f}(\xi + \tau) \right) d\xi \right) d\tau \\
 (3.5) \quad &= \int_{\mathbb{R}^n} e^{ix \cdot \tau} \sigma_\tau(X, D) \chi(D) f_\tau(x) d\tau \\
 &= \sum_{Q, Q' \in \mathcal{Q}} \int_{\mathbb{R}^n} e^{ix \cdot \tau} \sigma_{\tau, Q, Q'}(X, D) \chi(D) f_\tau(x) d\tau = \sum_{Q, Q' \in \mathcal{Q}} h_{Q, Q'}(x),
 \end{aligned}$$

where

$$h_{Q, Q'}(x) = \int_{\mathbb{R}^n} e^{ix \cdot \tau} g_{Q, Q'}(x, \tau) d\tau, \quad g_{Q, Q'}(x, \tau) = \sigma_{\tau, Q, Q'}(X, D) \chi(D) f_\tau(x).$$

We consider  $h_{Q, Q'}$ , and set  $(g_{Q, Q'})_\tau(x) = g_{Q, Q'}(x, \tau)$ . Since  $\text{supp } \psi_Q \subset Q$ ,  $\text{supp } \varphi \subset B(0, 1)$  and

$$\begin{aligned}
 (\widehat{g_{Q, Q'}})_\tau(y) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_{x \rightarrow y} [e^{ix \cdot \xi} \sigma_{\tau, Q, Q'}(x, \xi)] \chi(\xi) \varphi(\xi) \widehat{f}(\xi + \tau) d\xi \\
 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} [\mathcal{F}_1 \sigma_{\tau, Q, Q'}](y - \xi, \xi) \chi(\xi) \varphi(\xi) \widehat{f}(\xi + \tau) d\xi \\
 &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi_Q(y - \xi) [\mathcal{F}_1(\psi_{Q'}(D_\xi) \sigma_\tau)](y - \xi, \xi) \chi(\xi) \varphi(\xi) \widehat{f}(\xi + \tau) d\xi,
 \end{aligned}$$

we see that  $\text{supp } (\widehat{g_{Q, Q'}})_\tau \subset Q + B(0, 1)$ . On the other hand, it is easy to show that  $\sup_{x \in \mathbb{R}^n} \|g_{Q, Q'}(x, \cdot)\|_{L^1(\mathbb{R}^n)} < \infty$  since

$$(3.6) \quad \|\sigma_{\tau, Q, Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\mathcal{F}^{-1} \psi_Q\|_{L^1} \|\mathcal{F}^{-1} \psi_{Q'}\|_{L^1} \|\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)},$$



and  $g_{Q,Q'}(x, \tau) \in L^2(\mathbb{R}_x^n \times \mathbb{R}_\tau^n)$  will be proved in the below. Hence, by Lemma 3.2 and (2.3), we have

$$(3.7) \quad \|h_{Q,Q'}\|_{L^2} \leq C|Q+B(0,1)|^{1/2}\|g_{Q,Q'}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} \leq C|Q|^{1/2}\|g_{Q,Q'}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$$

for all  $Q, Q' \in \mathcal{Q}$ . We next consider  $g_{Q,Q'}$ , and set  $\widetilde{\sigma_{\tau,Q,Q'}}(x, \xi) = \sigma_{\tau,Q,Q'}(x, \xi)\chi(\xi)$  and  $(\widetilde{\sigma_{\tau,Q,Q'}})_x(\xi) = \widetilde{\sigma_{\tau,Q,Q'}}(x, \xi)$ . Then

$$(3.8) \quad g_{Q,Q'}(x, \tau) = \widetilde{\sigma_{\tau,Q,Q'}}(X, D)f_\tau(x).$$

Since  $\text{supp } \psi_{Q'} \subset Q'$ ,  $\text{supp } \widehat{\chi} \subset B(0, 1)$  and

$$\begin{aligned} \mathcal{F}[(\widetilde{\sigma_{\tau,Q,Q'}})_x](\eta) &= \frac{1}{(2\pi)^n}(\mathcal{F}_2\sigma_{\tau,Q,Q'}(x, \cdot)) * \widehat{\chi}(\eta) \\ &= \frac{1}{(2\pi)^n}(\psi_{Q'}(\mathcal{F}_2\psi_Q(D_x)\sigma_\tau)(x, \cdot)) * \widehat{\chi}(\eta), \end{aligned}$$

we see that  $\text{supp } \mathcal{F}[(\widetilde{\sigma_{\tau,Q,Q'}})_x] \subset Q' + B(0, 1)$ . On the other hand, (3.6) gives  $(\widetilde{\sigma_{\tau,Q,Q'}})_x(\xi) \in L^1(\mathbb{R}_\xi^n) \cap L^2(\mathbb{R}_\xi^n)$ . Thus, by (2.3), (3.8) and Lemma 3.3, we have

$$\begin{aligned} \|g_{Q,Q'}(\cdot, \tau)\|_{L^2} &\leq C|Q' + B(0, 1)|^{1/2} \sup_{x \in \mathbb{R}^n} \|\widetilde{\sigma_{\tau,Q,Q'}}(x, \cdot)\|_{L^2} \|f_\tau\|_{L^2} \\ &\leq C|Q'|^{1/2} \|\sigma_{\tau,Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|\chi\|_{L^2} \|f_\tau\|_{L^2} \\ &= C|Q'|^{1/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|f_\tau\|_{L^2} \end{aligned}$$

for all  $Q, Q' \in \mathcal{Q}$ . This implies

$$\begin{aligned} \|g_{Q,Q'}\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} &= \left\{ \int_{\mathbb{R}^n} \|g_{Q,Q'}(\cdot, \tau)\|_{L^2}^2 d\tau \right\}^{1/2} \\ (3.9) \quad &\leq C|Q'|^{1/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f_\tau(x)|^2 dx \right) d\tau \right\}^{1/2} \\ &= C|Q'|^{1/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |\widehat{f}_\tau(\xi)|^2 d\xi \right) d\tau \right\}^{1/2} \\ &= C|Q'|^{1/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|\varphi\|_{L^2} \|f\|_{L^2} \end{aligned}$$

for all  $Q, Q' \in \mathcal{Q}$ . Recall that  $\langle x_Q \rangle^{\alpha n} \asymp |Q|$  and  $\langle \xi_{Q'} \rangle^{\alpha n} \asymp |Q'|$  for all  $Q, Q' \in \mathcal{Q}$ , where  $x_Q \in Q$  and  $\xi_{Q'} \in Q'$  (see the definition of an  $\alpha$ -covering). Therefore, by (3.5), (3.7) and (3.9),

$$\begin{aligned} \|\sigma(X, D)f\|_{L^2} &= \|h\|_{L^2} \leq \sum_{Q, Q' \in \mathcal{Q}} \|h_{Q,Q'}\|_{L^2} \\ &\leq C \left( \sum_{Q, Q' \in \mathcal{Q}} |Q|^{1/2} |Q'|^{1/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \right) \|f\|_{L^2} \\ &\leq C \left( \sum_{Q, Q' \in \mathcal{Q}} \langle x_Q \rangle^{\alpha n/2} \langle \xi_{Q'} \rangle^{\alpha n/2} \|\psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \right) \|f\|_{L^2}. \end{aligned}$$

This is the desired result.

4. COMMUTATORS AND  $\alpha$ -MODULATION SPACES

In this section, we prove Theorem 1.2. We recall the definition of commutators. Let  $a$  be a Lipschitz function on  $\mathbb{R}^n$ , that is,

$$(4.1) \quad |a(x) - a(y)| \leq A|x - y| \quad \text{for all } x, y \in \mathbb{R}^n.$$

Note that  $a$  satisfies (4.1) if and only if  $a$  is differentiable (in the ordinary sense) and  $\partial^\beta a \in L^\infty(\mathbb{R}^n)$  for  $|\beta| = 1$  (see [23, Chapter 8, Theorem 3]). If  $T$  is a bounded linear operator on  $L^2(\mathbb{R}^n)$ , then  $T(af)$  and  $aTf$  make sense as elements in  $L^2_{\text{loc}}(\mathbb{R}^n)$  when  $f \in \mathcal{S}(\mathbb{R}^n)$ , since  $|a(x)| \leq C(1 + |x|)$  for some constant  $C > 0$ . Hence, the commutator  $[T, a]$  can be defined by

$$[T, a]f(x) = T(af)(x) - a(x)Tf(x) \quad \text{for } f \in \mathcal{S}(\mathbb{R}^n).$$

In order to prove Theorem 1.2, we prepare the following lemmas:

**Lemma 4.1.** *Let  $T$  be a bounded linear operator on  $L^2(\mathbb{R}^n)$ , and  $a$  be a Lipschitz function on  $\mathbb{R}^n$  with  $\|\nabla a\|_{L^\infty} \neq 0$ . Then there exist  $\epsilon(a) > 0$  and  $\{a_\epsilon\}_{0 < \epsilon < \epsilon(a)} \subset \mathcal{S}(\mathbb{R}^n)$  such that*

- (1)  $\langle [T, a]f, g \rangle = \lim_{\epsilon \rightarrow 0} \langle [T, a_\epsilon]f, g \rangle$  for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$ ,
- (2)  $\|\nabla a_\epsilon\|_{L^\infty} \leq C\|\nabla a\|_{L^\infty}$  for all  $0 < \epsilon < \epsilon(a)$ ,

where  $C$  is independent of  $T$  and  $a$ ,  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$ -inner product, and  $\nabla a = (\partial_1 a, \dots, \partial_n a)$ .

*Proof.* Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  be such that  $\varphi(0) = 1$ ,  $\int_{\mathbb{R}^n} \varphi(x) dx = 1$  and  $\text{supp } \varphi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ . If we set  $a_\epsilon(x) = \varphi(\epsilon x)(\varphi_\epsilon * a)(x)$ , then  $\{a_\epsilon\}_{0 < \epsilon < \epsilon(a)} \subset \mathcal{S}(\mathbb{R}^n)$  satisfies (1) and (2), where  $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$  and  $\epsilon(a)$  will be chosen in the below.

We first consider (2). Since  $|a(x) - a(y)| \leq \|\nabla a\|_{L^\infty}|x - y|$  for all  $x, y \in \mathbb{R}^n$ , we see that

$$\begin{aligned} |\partial_i(a_\epsilon(x))| &\leq \epsilon|(\partial_i \varphi)(\epsilon x) \varphi_\epsilon * a(x)| + |\varphi(\epsilon x) \varphi_\epsilon * (\partial_i a)(x)| \\ &\leq \epsilon|(\partial_i \varphi)(\epsilon x) (\varphi_\epsilon * a(x) - a(0))| + \epsilon|(\partial_i \varphi)(\epsilon x) a(0)| + \|\varphi\|_{L^1} \|\varphi\|_{L^\infty} \|\nabla a\|_{L^\infty} \\ &\leq \epsilon|(\nabla \varphi)(\epsilon x)| \int_{\mathbb{R}^n} \|\nabla a\|_{L^\infty} (1 + |x|)(1 + \epsilon|y|)|\varphi(y)| dy \\ &\quad + \epsilon|a(0)| \|\nabla \varphi\|_{L^\infty} + \|\varphi\|_{L^1} \|\varphi\|_{L^\infty} \|\nabla a\|_{L^\infty} \\ &\leq C_\varphi^1 C_\varphi^2 \|\nabla a\|_{L^\infty} + \epsilon|a(0)| \|\nabla \varphi\|_{L^\infty} + \|\varphi\|_{L^1} \|\varphi\|_{L^\infty} \|\nabla a\|_{L^\infty} \end{aligned}$$

for all  $0 < \epsilon < 1$ , where  $C_\varphi^1 = \sup_{x \in \mathbb{R}^n} (1 + |x|)|\nabla \varphi(x)|$  and  $C_\varphi^2 = \int_{\mathbb{R}^n} (1 + |y|)|\varphi(y)| dy$ . Hence, we obtain (2) with  $\epsilon(a) = \min\{\|\nabla a\|_{L^\infty}/|a(0)|, 1\}$  if  $a(0) \neq 0$ , and  $\epsilon(a) = 1$  if  $a(0) = 0$ .

We next consider (1). Since  $a$  is continuous and  $|a(x)| \leq C(1 + |x|)$  for all  $x \in \mathbb{R}^n$ , we see that  $\lim_{\epsilon \rightarrow 0} a_\epsilon(x) = a(x)$  for all  $x \in \mathbb{R}^n$ , and  $|a_\epsilon(x)| \leq C\|\varphi\|_{L^\infty} C_\varphi^2(1 + |x|)$  for all  $0 < \epsilon < \epsilon(a)$  and  $x \in \mathbb{R}^n$ . Hence, by the Lebesgue dominated convergence theorem, we have that  $\lim_{\epsilon \rightarrow 0} \langle a_\epsilon T f, g \rangle = \langle a T f, g \rangle$  for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$ , and  $a_\epsilon f \rightarrow af$  in  $L^2(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$  for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , and consequently  $T(a_\epsilon f) \rightarrow T(af)$  in  $L^2(\mathbb{R}^n)$  as  $\epsilon \rightarrow 0$  for all  $f \in \mathcal{S}(\mathbb{R}^n)$ . The proof is complete.  $\square$

**Lemma 4.2.** *Let  $\sigma(x, \xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  be such that  $\text{supp } \widehat{\sigma_x} \subset \Omega$ , where  $\sigma_x(\xi) = \sigma(x, \xi)$ ,  $\widehat{\sigma_x}(\eta) = \mathcal{F}_2 \sigma(x, \eta)$  and  $\Omega$  is a compact subset of  $\mathbb{R}^n$  independent of  $x$ . Then there exists a constant  $C > 0$  such that*

$$|\sigma(X, D)f(x)| \leq C|\Omega|^{1/2} \|\sigma(x, \cdot)\|_{L^2} \|f\|_{L^\infty}$$

for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , where  $C$  is independent of  $\sigma$  and  $\Omega$ .

*Proof.* Since

$$\begin{aligned}\sigma(X, D)f(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{-iy \cdot \xi} \sigma(x, \xi) d\xi \right) f(x+y) dy \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\sigma}_x(y) f(x+y) dy = \frac{1}{(2\pi)^n} \int_{\Omega} \widehat{\sigma}_x(y) f(x+y) dy,\end{aligned}$$

we have by Schwartz's inequality and Plancherel's theorem

$$|\sigma(X, D)f(x)| \leq C_n |\Omega|^{1/2} \|\widehat{\sigma}_x\|_{L^2} \|f\|_{L^\infty} = C_n |\Omega|^{1/2} \|\sigma_x\|_{L^2} \|f\|_{L^\infty}.$$

The proof is complete.  $\square$

**Lemma 4.3.** *Let  $0 \leq \alpha \leq 1$ . Then there exists an  $\alpha$ -covering  $\mathcal{Q}$  of  $\mathbb{R}^n$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$  satisfying*

$$\|\partial^\beta(\mathcal{F}^{-1}\psi_Q)\|_{L^1} \leq C_\beta \langle \xi_Q \rangle^{|\beta|} \quad \text{for all } \xi_Q \in Q \text{ and } Q \in \mathcal{Q},$$

where  $\beta \in \mathbb{Z}_+^n = \{0, 1, \dots\}^n$ .

*Proof.* If  $\alpha = 1$  then Lemma 4.3 is trivial, since we can take  $\{\varphi_j\}_{j \geq 0}$  as a BAPU corresponding to the  $\alpha$ -covering  $\{\{|\xi| \leq 2\}, \{\{2^{j-1} \leq |\xi| \leq 2^{j+1}\}\}_{j \geq 1}\}$ , where  $\{\varphi_j\}_{j \geq 0}$  is as in (2.7).

We consider the case  $0 \leq \alpha < 1$ . Let  $B_k^r = B(|k|^{\alpha/(1-\alpha)}k, r|k|^{\alpha/(1-\alpha)})$  and  $\Phi \in \mathcal{S}(\mathbb{R}^n)$  be such that  $\inf_{|\xi| \leq r/2} |\Phi(\xi)| > 0$  and  $\text{supp } \Phi \subset B(0, r)$ , where  $k \in \mathbb{Z}^n \setminus \{0\}$  and  $r$  is sufficiently large. Set

$$\psi_k(\xi) = \frac{g_k(\xi)}{\sum_{n \in \mathbb{Z}^n \setminus \{0\}} g_n(\xi)} \quad \text{and} \quad g_k(\xi) = \Phi(|c_k|^{-\alpha}(\xi - c_k)), \quad k \in \mathbb{Z}^n \setminus \{0\},$$

where  $c_k = |k|^{\alpha/(1-\alpha)}k$ . In the proof of [1, Proposition A.1] (or [2, Proposition 2.4]), Borup and Nielsen proved that the pair of  $\{B_k^r\}_{k \in \mathbb{Z}^n \setminus \{0\}}$  and  $\{\psi_k\}_{k \in \mathbb{Z}^n \setminus \{0\}}$  is an  $\alpha$ -covering of  $\mathbb{R}^n$  with a corresponding BAPU, and  $|\partial^\beta \psi_k(\xi)| \leq C_\beta \langle \xi \rangle^{-|\beta|\alpha}$  and  $\|\partial^\beta \widetilde{\psi}_k\|_{L^1} \leq C'_\beta$  for all  $k \in \mathbb{Z}^n \setminus \{0\}$  and  $\beta \in \mathbb{Z}_+^n$ , where  $\widetilde{\psi}_k(\xi) = \psi(|c_k|^\alpha \xi + c_k)$ . Since  $\{B_k^r\}_{k \in \mathbb{Z}^n \setminus \{0\}}$  is an  $\alpha$ -covering of  $\mathbb{R}^n$ , we have  $\langle c_k \rangle \asymp \langle \xi_{B_k^r} \rangle$  for all  $\xi_{B_k^r} \in B_k^r$  and  $k \in \mathbb{Z}^n \setminus \{0\}$ . Noting  $\text{supp } \widetilde{\psi}_k \subset B(0, r)$ , we see that

$$\begin{aligned}\|\partial^\beta(\mathcal{F}^{-1}\psi_k)\|_{L^1} &= \int_{\mathbb{R}^n} \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\beta \psi_k(\xi) d\xi \right| dx \\ &= \int_{\mathbb{R}^n} \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|c_k|^\alpha \xi + c_k)^\beta \widetilde{\psi}_k(\xi) d\xi \right| dx \\ &\leq C_\beta \langle c_k \rangle^{|\beta|} \left( \sum_{|\gamma| \leq n+1} \|\partial^\gamma \widetilde{\psi}_k\|_{L^1} \right) \int_{\mathbb{R}^n} \langle x \rangle^{-n-1} dx \leq C_{\beta, n} \langle \xi_{B_k^r} \rangle^{|\beta|}\end{aligned}$$

for all  $\xi_{B_k^r} \in B_k^r$ ,  $k \in \mathbb{Z}^n \setminus \{0\}$  and  $\beta \in \mathbb{Z}_+^n$ . The proof is complete.  $\square$

We are now ready to prove Theorem 1.2.

*Proof of Theorem 1.2.* Let  $\sigma \in M_{(\alpha n/2, \alpha n+1), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $a$  be a Lipschitz function on  $\mathbb{R}^n$ . Then, by Theorem 1.1, we see that  $\sigma(X, D)$  is bounded on  $L^2(\mathbb{R}^n)$ .

Since  $[\sigma(X, D), a] = 0$  if  $a$  is a constant function, we may assume  $\|\nabla a\|_{L^\infty} \neq 0$ . Hence, by Lemmas 3.4 and 4.1, we have

$$\langle [\sigma(X, D), a]f, g \rangle = \lim_{\epsilon \rightarrow 0} \langle [\sigma(X, D), a_\epsilon]f, g \rangle = \lim_{\epsilon \rightarrow 0} \left( \lim_{\epsilon' \rightarrow 0} \langle [\sigma_{\epsilon'}(X, D), a_\epsilon]f, g \rangle \right)$$

for all  $f, g \in \mathcal{S}(\mathbb{R}^n)$ , where  $\{\sigma_{\epsilon'}\}_{0 < \epsilon' < 1} \subset \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $\{a_\epsilon\}_{0 < \epsilon < \epsilon(a)} \subset \mathcal{S}(\mathbb{R}^n)$  are as in Lemmas 3.4 and 4.1. Hence, it is enough to prove Theorem 1.2 with  $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $a \in \mathcal{S}(\mathbb{R}^n)$ . We note that

(4.2)

$$[\sigma(X, D), a]f(x) = C_n \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \int_{\mathbb{R}^n} e^{ix \cdot \eta} (\sigma(x, \xi + \eta) - \sigma(x, \xi)) \widehat{a}(\eta) d\eta \right) \widehat{f}(\xi) d\xi$$

for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , where  $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$  and  $a \in \mathcal{S}(\mathbb{R}^n)$ . In fact,

$$\begin{aligned} \sigma(X, D)(af)(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \sigma(x, \eta) \widehat{af}(\eta) d\eta \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \sigma(x, \eta) \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{a}(\eta - \xi) \widehat{f}(\xi) d\xi \right) d\eta \end{aligned}$$

and

$$a(x)\sigma(X, D)f(x) = \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \eta} \widehat{a}(\eta) d\eta \right) \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) d\xi.$$

We decompose  $\sigma$  and  $a$  as follows:

$$(4.3) \quad \sigma(x, \xi) = \sum_{Q, Q' \in \mathcal{Q}} \sigma_{Q, Q'}(x, \xi) \quad \text{and} \quad a(x) = \sum_{j=0}^{\infty} \varphi_j(D)a(x),$$

where  $\sigma_{Q, Q'}(x, \xi) = \psi_Q(D_x)\psi_{Q'}(D_\xi)\sigma(x, \xi)$ ,  $\mathcal{Q}$  is an  $\alpha$ -covering of  $\mathbb{R}^n$  with a corresponding BAPU  $\{\psi_Q\}_{Q \in \mathcal{Q}} \subset \mathcal{S}(\mathbb{R}^n)$ , and  $\{\varphi_j\}_{j \geq 0}$  is as in (2.7). Then, by the decomposition (4.3),

$$(4.4) \quad [\sigma(X, D), a] = \sum_{Q, Q' \in \mathcal{Q}} [\sigma_{Q, Q'}(X, D), \varphi_0(D)a] + \sum_{j=1}^{\infty} [\sigma(X, D), \varphi_j(D)a].$$

We consider the first sum of the right-hand side of (4.4). By (4.2) and Taylor's formula, we have

$$\begin{aligned} &[\sigma_{Q, Q'}(X, D), \varphi_0(D)a]f(x) \\ &= C_n \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left\{ \int_{\mathbb{R}^n} e^{ix \cdot \eta} \left( \sum_{k=1}^n \eta_k \int_0^1 \partial_{\xi_k} \sigma_{Q, Q'}(x, \xi + t\eta) dt \right) \varphi_0(\eta) \widehat{a}(\eta) d\eta \right\} \widehat{f}(\xi) d\xi \\ &= C_n \sum_{k=1}^n \int_0^1 \left\{ \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left( \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q, Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right) \widehat{f}(\xi) d\xi \right\} dt, \end{aligned}$$

where  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ . Hence, by Theorem 1.1,

$$\begin{aligned} &\|[\sigma_{Q, Q'}(X, D), \varphi_0(D)a]f\|_{L^2} \\ &\leq C \|f\|_{L^2} \sum_{k=1}^n \int_0^1 \\ &\quad \times \left\| \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q, Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right\|_{M_{(\alpha n/2, \alpha n/2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} dt. \end{aligned} \tag{4.5}$$

Note that  $\partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \in \mathcal{S}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)$ . Since

$$\begin{aligned} \mathcal{F}_{x \rightarrow y} \left[ \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right] &\subset \{y \in \mathbb{R}^n : y \in Q + \overline{B(0, 2)}\}, \\ \mathcal{F}_{\xi \rightarrow \zeta} \left[ \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right] &\subset \{\zeta \in \mathbb{R}^n : \zeta \in Q'\} \end{aligned}$$

and  $\sup_{Q \in \mathcal{Q}} \|\mathcal{F}^{-1} \psi_Q\|_{L^1} < \infty$ , we have by Lemma 2.1

$$\begin{aligned} (4.6) \quad & \left\| \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right\|_{M_{(\alpha n/2, \alpha n/2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \\ &= \sum_{\substack{\tilde{Q} \cap (Q + \overline{B(0, 2)}) \neq \emptyset \\ \tilde{Q} \in \mathcal{Q}}} \sum_{\substack{\tilde{Q}' \cap Q' \neq \emptyset \\ \tilde{Q}' \in \mathcal{Q}}} \langle x_{\tilde{Q}} \rangle^{\alpha n/2} \langle \xi_{\tilde{Q}'} \rangle^{\alpha n/2} \\ &\quad \times \left\| \psi_{\tilde{Q}}(D_x) \psi_{\tilde{Q}'}(D_\xi) \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &\leq C \langle x_Q \rangle^{\alpha n/2} \langle \xi_{Q'} \rangle^{\alpha n/2} \left\| \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}. \end{aligned}$$

Let  $\chi \in \mathcal{S}(\mathbb{R}^n)$  be such that  $|\chi| \geq 1$  on  $\{|\xi| \leq 4\}$  and  $\text{supp } \widehat{\chi} \subset \{|x| < 1\}$  (for the existence of such a function, see the proof of [14, Theorem 2.6]). Since  $\varphi_0 = \varphi_0 \chi / \chi = \chi (\varphi_0 / \chi)$ , we can write  $\varphi_0 = \chi \Phi$ , where  $\Phi = \varphi_0 / \chi \in \mathcal{S}(\mathbb{R}^n)$ . Then

$$\begin{aligned} (4.7) \quad & \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \\ &= \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \chi(\eta) \Phi(\eta) \widehat{\partial_k a}(\eta) d\eta = \tau_{Q,Q'}^{k,t,\xi}(X, D)(\Phi(D)(\partial_k a))(x), \end{aligned}$$

where  $\tau_{Q,Q'}^{k,t,\xi}(x, \eta) = \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \chi(\eta)$ . Since

$$\mathcal{F}_{\eta \rightarrow \zeta} [\partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta)] = t^{-n} (i\zeta_k / t) e^{i\xi \cdot \zeta / t} \psi_{Q'}(\zeta / t) \mathcal{F}_2[\psi_Q(D_x) \sigma](x, \zeta / t),$$

we have

$$(4.8) \quad \text{supp } \mathcal{F}[(\tau_{Q,Q'}^{k,t,\xi})_x] \subset \{\zeta \in \mathbb{R}^n : \zeta \in tQ' + B(0, 1)\},$$

where  $(\tau_{Q,Q'}^{k,t,\xi})_x(\eta) = \tau_{Q,Q'}^{k,t,\xi}(x, \eta)$  and  $tQ' = \{t\zeta : \zeta \in Q'\}$ . On the other hand, by (2.5), (2.8) and Lemma 4.3, we see that

$$\begin{aligned} (4.9) \quad & \|\partial_{\xi_k} \sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \sum_{\tilde{Q}' \in \mathcal{Q}} \|\partial_{\xi_k} (\psi_{\tilde{Q}'}(D_\xi) \sigma_{Q,Q'})\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \\ &= \sum_{\tilde{Q}' \cap Q' \neq \emptyset} \sup_{x \in \mathbb{R}^n} \left\| [\partial_{\xi_k} (\mathcal{F}^{-1} \psi_{\tilde{Q}'})] * \sigma_{Q,Q'}(x, \cdot) \right\|_{L^\infty} \\ &\leq \sum_{\tilde{Q}' \cap Q' \neq \emptyset} \sup_{x \in \mathbb{R}^n} \|\partial_{\xi_k} (\mathcal{F}^{-1} \psi_{\tilde{Q}'})\|_{L^1} \|\sigma_{Q,Q'}(x, \cdot)\|_{L^\infty} \\ &\leq C \sum_{\tilde{Q}' \cap Q' \neq \emptyset} \langle \xi_{\tilde{Q}'} \rangle \|\sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq C n_0 \langle \xi_{Q'} \rangle \|\sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}. \end{aligned}$$

We note that  $\tau_{Q,Q'}^{k,t,\xi}(x,\eta) \in \mathcal{S}(\mathbb{R}_x^n \times \mathbb{R}_\eta^n)$  for every  $1 \leq k \leq n$ ,  $0 < t < 1$  and  $\xi \in \mathbb{R}^n$ , since  $\sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ . Thus, by (2.3), (4.7), (4.8), (4.9) and Lemma 4.2, we obtain that

$$\begin{aligned}
 (4.10) \quad & \sup_{x,\xi \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{ix \cdot \eta} \partial_{\xi_k} \sigma_{Q,Q'}(x, \xi + t\eta) \varphi_0(\eta) \widehat{\partial_k a}(\eta) d\eta \right| \\
 & \leq C|tQ' + B(0,1)|^{1/2} \left( \sup_{x,\xi \in \mathbb{R}^n} \|\tau_{Q,Q'}^{k,t,\xi}(x, \cdot)\|_{L^2} \right) \|\Phi(D)(\partial_k a)\|_{L^\infty} \\
 & \leq C|Q'|^{1/2} \|\partial_{\xi_k} \sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|\chi\|_{L^2} \|\Phi\|_{L^1} \|\partial_k a\|_{L^\infty} \\
 & \leq C\langle \xi_{Q'} \rangle^{\alpha n/2+1} \|\sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \|\nabla a\|_{L^\infty}
 \end{aligned}$$

for all  $0 < t < 1$ . Combining (4.5), (4.6) and (4.10), we have

$$\begin{aligned}
 & \|[\sigma(X, D), \varphi_0(D)a]f\|_{L^2} \leq \sum_{Q,Q' \in \mathcal{Q}} \|[\sigma_{Q,Q'}(X, D), \varphi_0(D)a]f\|_{L^2} \\
 & \leq C\|\nabla a\|_{L^\infty} \left( \sum_{Q,Q' \in \mathcal{Q}} \langle x_Q \rangle^{\alpha n/2} \langle \xi_{Q'} \rangle^{\alpha n+1} \|\sigma_{Q,Q'}\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \right) \|f\|_{L^2} \\
 & = C\|\nabla a\|_{L^\infty} \|\sigma\|_{M_{(\alpha n/2, \alpha n+1), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \|f\|_{L^2}.
 \end{aligned}$$

We next consider the second sum of the right-hand side (4.4). Since

$$\varphi_j(D)a(x) = \int_{\mathbb{R}^n} 2^{jn} (\mathcal{F}^{-1}\varphi)(2^j(x-y)) (a(y) - a(x)) dx$$

and  $a$  is a Lipschitz function, we have  $\|\varphi_j(D)a\|_{L^\infty} \leq C2^{-j}\|\nabla a\|_{L^\infty}$  for all  $j \geq 1$ . Hence, by Theorem 1.1, we see that

$$\begin{aligned}
 & \sum_{j=1}^{\infty} \|[\sigma(X, D), \varphi_j(D)a]f\|_{L^2} \\
 & \leq \sum_{j=1}^{\infty} (\|\sigma(X, D)(\varphi_j(D)a)f\|_{L^2} + \|(\varphi_j(D)a)\sigma(X, D)f\|_{L^2}) \\
 & \leq C \sum_{j=1}^{\infty} 2^{-j} \|\nabla a\|_{L^\infty} \|\sigma\|_{M_{(\alpha n/2, \alpha n/2), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \|f\|_{L^2} \\
 & \leq C\|\nabla a\|_{L^\infty} \|\sigma\|_{M_{(\alpha n/2, \alpha n+1), (\alpha, \alpha)}^{(\infty, \infty), (1, 1)}} \|f\|_{L^2}.
 \end{aligned}$$

The proof is complete.

#### APPENDIX A. THE INCLUSION BETWEEN BESOV AND MODULATION SPACES

Let  $1 \leq p, q \leq \infty$  and  $p'$  be the conjugate exponent of  $p$  (that is,  $1/p + 1/p' = 1$ ). In [26, Theorem 3.1], Toft proved the inclusions

$$B_{n\nu_1(p,q)}^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n) \hookrightarrow B_{n\nu_2(p,q)}^{p,q}(\mathbb{R}^n),$$

where

$$\begin{aligned}
 \nu_1(p, q) &= \max\{0, 1/q - \min(1/p, 1/p')\}, \\
 \nu_2(p, q) &= \min\{0, 1/q - \max(1/p, 1/p')\}.
 \end{aligned}$$

Due to Sugimoto-Tomita [25, Theorem 1.2], the optimality of the inclusion relation between Besov and modulation spaces is described in the following way:

**Theorem A.1.** *Let  $1 \leq p, q \leq \infty$  and  $s \in \mathbb{R}$ . Then the following are true:*

- (1) *If  $B_s^{p,q}(\mathbb{R}^n) \hookrightarrow M^{p,q}(\mathbb{R}^n)$ , then  $s \geq n\nu_1(p, q)$ .*
- (2) *If  $M^{p,q}(\mathbb{R}^n) \hookrightarrow B_s^{p,q}(\mathbb{R}^n)$ , then  $s \leq n\nu_2(p, q)$ .*

In particular, we have the best inclusions

$$B_n^{\infty,1}(\mathbb{R}^n) \hookrightarrow M^{\infty,1}(\mathbb{R}^n) \hookrightarrow B_0^{\infty,1}(\mathbb{R}^n).$$

Hence, we see that  $B_{n/2}^{\infty,1}(\mathbb{R}^n)$  and  $M^{\infty,1}(\mathbb{R}^n)$  have no inclusion relation with each others. We remark that the statement (2) was shown in a restricted case  $1 \leq p, q < \infty$  in [25], but it is also true for the endpoint  $p = \infty$  or  $q = \infty$ . For example, if we assume that  $M^{\infty,q}(\mathbb{R}^n) \hookrightarrow B_s^{\infty,q}(\mathbb{R}^n)$  with  $s > n\nu_2(\infty, q)$ , then we have  $M^{p,\tilde{q}}(\mathbb{R}^n) \hookrightarrow B_s^{p,\tilde{q}}(\mathbb{R}^n)$  ( $2 < p < \infty$ ) with  $s > n\nu_2(p, \tilde{q})$  by interpolating it with the fact  $M^{2,2} = B_0^{2,2}$ , where  $1 < \tilde{q} < \infty$  is a number determined by  $p$  and  $q$ . This contradicts to (2) with  $1 \leq p, q < \infty$ .

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